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Note

A result on Vizing's conjecture

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Abstract

Let $\gamma(G)$ denote the domination number of a simple graph G and let $G\Box H$ denote the Cartesian product of two simple graphs G and H. In this paper we prove that if $\gamma(G)=3$, then $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

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1. Introduction

For a simple graph G, a subset S of $V(G)$ is a *dominating set* of G if for any vertex $v \in V(G) - S$, there exists a vertex $v' \in S$ such that v and v' are adjacent in G. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For a subset S of $V(G)$, we define $N_G(S) = \{v \in V(G) - S : v$ is adjacent to a vertex in $S\}$ and $N_G[S] = S \cup N_G(S)$. Then S is a dominating set of G if and only if $N_G[S] = V(G)$.

Let G and H be two simple graphs. The *Cartesian product* $G\Box H$ of G and H has $V(G\Box H) = V(G) \times V(H)$ and vertices (u_1, v_1) and (u_2, v_2) in $V(G\Box H)$ are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

In 1963, Vizing [4] conjectured that for any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$. For $\gamma(G)$=1 and 2, it is easy to verify the validity of Vizing's conjecture [3]. Brešar [2] proved that Vizing's conjecture is true when $\gamma(G) = \gamma(H) = 3$. But for the general graph H, Vizing's conjecture for the case $\gamma(G) = 3$ is still open [3]. In this paper we

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are going to settle this case and prove that Vizing's conjecture is true for $\gamma(G)=3$. For the other progress on Vizing's conjecture, the readers may refer to the survey paper of Hartnell and Rall [3].

2. Main results

In this section we will prove that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$, when $\gamma(G)=3$. For this purpose we need a classical theorem about bipartite graphs.

Theorem 1 (Bondy and Murty [1]). If G is a bipartite graph, then the minimum number of vertices in a covering of G is equal to the maximum number of edges in a matching of G.

Next we give some notation. Denote $V(G)=\{1,\dots,m\}$, $V(H)=\{1,\dots,n\}$, and $V(G\Box H)=\{(i,j) : i=1,\dots,m; j=1,\dots,n\}$. Let $U_j=\{(i,j) : i=1,\dots,m\}$, and G_j be the subgraph of $G\Box H$ induced by U_j, $j=1,\dots,n$. Let $V_i=\{(i,j) : j=1,\dots,n\}$, and H_i be the subgraph of $G\Box H$ induced by V_i, $i=1,\dots,m$. For any vertex (i,j) of $G\Box H$, the vertex j of H is the H-*projection* of (i,j), denoted $j=\phi_H(i,j)$. For any subset $A=\{(i_1,j_1),\dots,(i_k,j_k)\}$ of $V(G\Box H)$, we denote $\phi_H(A)=\{j_1,\dots,j_k\}$ which is a subset of $V(H)$. For a subset A of $V(G\Box H)$, if $\phi_H(A)=V(H)$, then A is H-*full*. Obviously A is H-full if and only if $A\cap U_j \neq \emptyset$ for $j=1,2,\dots,n$. For simplicity, we denote $N_{G\Box H}(A)=N(A)$ and $N_{G\Box H}[A]=N[A]$, where A is a subset of $V(G\Box H)$.

Now we prove the main result of this paper.

Theorem 2. If $\gamma(G)=3$, then for any graph H, $\gamma(G\Box H)\geq\gamma(G)\gamma(H)$.

Proof. Let G be a graph with domination number 3. Without loss of generality, we assume that G is critical, that is, for any two non-adjacent vertices u and v of G, $\gamma(G+uv)=2$. Let $\{1,p,q\}$ be a dominating set of G such that $N_G[p]\cup N_G[q]=V(G)-\{1\}$. Then the vertex 1 of G can not be adjacent to p nor q. Let D be a dominating set of $G\Box H$. We are going to prove that $|D|\geq 3\gamma(H)$.

Denote $A_1=V_1\cap D$, $A_2=V_1\cap N(A_1)$, and $A_3=V_1-A_1-A_2$. Let A_4 be a maximal independent set of the subgraph of $G\Box H$ induced by A_3. Since $A_4\cap(A_1\cup A_2)=\emptyset$, for each vertex $(1,j)\in A_4$, there exists a vertex $(i_j,j)\in D\cap U_j$ such that $(1,j)$ and (i_j,j) are adjacent in $G\Box H$. Denote $A_5=\{(i_j,j) : j\in\phi_H(A_4)\}$, and $D_1=A_1\cup A_5$. Since $A_1\cup A_4$ dominates all vertices in H_1, and $|A_5|=|A_4|$, we have that $|D_1|\geq\gamma(H)$.

Since $\gamma(G)$=3, for any $k\in\phi_H(N(A_5)\cap V_p)$, there exists a vertex (t_k,k) in $N[(p,k)]\cap U_k$ such that (t_k,k) is not adjacent to (i_k,k) nor (q,k). Otherwise, $\{(i_k,k),(q,k)\}$ is a dominating set of G_k, which contradicts the fact that $\gamma(G)=3$. Denote $C_1=\{(t_k,k) : k\in\phi_H(N(A_5)\cap V_p)\}$, $C_2=\{(p,k) : k\notin\phi_H(N(A_5)\cap V_p)\}$, and $F=C_1\cup C_2$. Then $F\cap N[D_1]=\emptyset$, $F\subseteq N[V_p]$, and F is H-full. Denote $L_1=V_p\cap D$. Let $Y=\{(i,j)\in C_1-D : (i,j)$ can be dominated by $D\cap V_i$ and $j\notin\phi_H(N[L_1])\}$. Thus, for any vertex (i,j) in Y, there exists a vertex (i,j') in $D\cap V_i$ such that (i,j) and (i,j') are adjacent. Denote $L_2=\{(i,j')\in D : (i,j)\in Y\}$.

Now we discuss two cases according as p and q are adjacent or not in G.

Case 1: p and q are non-adjacent in G.

Let I be a maximal independent set of the subgraph of H induced by $V(H) - \phi_H(N[L_1 \cup L_2])$. Denote $Q = \{(i,j) \in F : j \in I\}$. Then for any vertex (i,j) in Q, there exists a vertex (i',j) in $U_j \cap D$ such that (i,j) and (i',j) are adjacent. Denote $L_3 = \{(i',j) \in D : (i,j) \in Q\}$.

Denote $D_2 = L_1 \cup L_2 \cup L_3$. Obviously $\phi_H(D_2)$ is a dominating set of H. Thus $|D_2| \geq \gamma(H)$. Since $F \cap N[D_1] = \emptyset$, we have that $D_1 \cap D_2 = \emptyset$.

Now we show that for each $j = 1, 2, \ldots, n$, $U_j - N[D_1 \cup D_2] \neq \emptyset$. By the definition of D_1 and D_2, we have that $0 \leq |U_j \cap (D_1 \cup D_2)| \leq 2$.

Case 1.1: $|U_j \cap (D_1 \cup D_2)| = 0$.

Since $(D_1 \cup D_2) \cap V_q = \emptyset$, we have that $(q,j) \notin N[D_1 \cup D_2]$.

Case 1.2: $|U_j \cap (D_1 \cup D_2)| = 1$.

Denote $U_j \cap (D_1 \cup D_2) = \{(i,j)\}$. If $(i,j) \in A_1 \cup L_1 \cup L_2$, then $(q,j) \notin N[D_1 \cup D_2]$. If $(i,j) \in A_5$, then we have that $U_j \cap N[D_1 \cup D_2] \subseteq N[(i,j),(p,j)]$, since $\phi_H(A_5)$ is independent in H. Thus we have that $U_j - N[D_1 \cup D_2] \neq \emptyset$.

Case 1.3: $|U_j \cap (D_1 \cup D_2)| = 2$.

Since $\phi_H(A_1 \cup A_5) \cap \phi_H(L_2) = \emptyset$, we have $j \in (\phi_H(A_1) \cap \phi_H(L_1)) \cup (\phi_H(A_1) \cap \phi_H(L_3)) \cup (\phi_H(A_5) \cap \phi_H(L_1)) \cup (\phi_H(A_5) \cap \phi_H(L_3))$. It is easy to verify that $U_j \cap N[D_1 \cup D_2] \subseteq N[U_j \cap (D_1 \cup D_2)]$. Thus we have that $U_j - N[D_1 \cup D_2] \neq \emptyset$, since $\gamma(G) = 3$.

Thus, for each $j = 1, 2, \ldots, n$, there exists a vertex (x_j, j) in U_j such that (x_j, j) is not dominated by $D_1 \cup D_2$. Denote $W_1 = \{(x_j, j) \in U_j - N[D_1 \cup D_2] : j = 1, 2, \ldots, n\}$. Since W_1 is H-full and W_1 must be dominated by $D_3 = D - D_1 - D_2$, we have that $|D_3| \geq \gamma(H)$. Therefore, $|D| = |D_1| + |D_2| + |D_3| \geq 3\gamma(H)$.

Case 2: p and q are adjacent in G.

If there is only one vertex r in $N_G[q] - (N_G[1] \cup N_G[p])$, then 1, p and r consist of a dominating set of G and the conclusion is true by the proof in Case 1. Thus we can assume that there are two distinct vertices s and t in $N_G[q] - (N_G[1] \cup N_G[p])$.

Denote $S_1 = \phi_H(C_2) - \phi_H(N[L_1 \cup L_2])$. Let I' be a maximal independent set of the subgraph of H induced by S_1. For every $j \in I'$, (p,j) must be dominated by a vertex (p',j) in $U_j \cap D$. Denote $L_4 = \{(p',j) : j \in I'\}$, and $S_2 = \phi_H(C_1) - \phi_H(N[L_1 \cup L_2 \cup L_4])$. Since $\phi_H(A_5)$ is independent in H, S_2 is independent in H also. Denote $T = \{(i,j) \in C_1 : j \in S_2\}$. Then for any $(i,j) \in T$, (i,j) must be dominated by a vertex (i',j) in $U_j \cap D$. Denote $T' = \{(i',j) : j \in S_2\}$, $T_1 = V_s \cap T'$ and $T_2 = V_t \cap T'$. Let B be a bipartite graph such that $V(B) = T_1 \cup T_2$, and a vertex $(s, j_1) \in T_1$ and a vertex $(t, j_2) \in T_2$ are adjacent in B if and only if there exists $j \in V(H)$ such that j is adjacent to both j_1 and j_2 in H. Let B_1 be the union of non-trivial components of B. By Theorem 1, we assume that $\{\alpha_1, \ldots, \alpha_\theta\}$ is a minimum vertex cover of B_1 and $\{\alpha_1\beta_1, \ldots, \alpha_\theta\beta_\theta\}$ is a maximum matching of B_1. Since there exists a vertex k_i in H such that k_i is adjacent to both $\phi_H(\alpha_i)$ and $\phi_H(\beta_i)$, $i = 1, 2, \ldots, \theta$, $\phi_H(T')$ can be dominated by $Z = \phi_H(T' - \{\alpha_1, \ldots, \alpha_\theta, \beta_1, \ldots, \beta_\theta\}) \cup \{k_1, \ldots, k_\theta\}$ in H. Thus $J = \phi_H(L_1 \cup L_2 \cup L_4) \cup Z$ is a dominating set of H. Denote $L_5 = T' - \{\alpha_1, \ldots, \alpha_\theta\}$, and $D_4 = L_1 \cup L_2 \cup L_4 \cup L_5$. Then D_4 is a subset of $D - D_1$ and $|D_4| = |J| \geq \gamma(H)$. Now we show that for each $j = 1, 2, \ldots, n$, $U_j - N[D_1 \cup D_4] \neq \emptyset$.

Case 2.1: $|U_j \cap (D_1 \cup D_4)| = 0$.

In this case, we have that either (s, j) or (t, j) can not be dominated by $D_1 \cup D_4$. Otherwise, we assume that both (s, j) and (t, j) are dominated by $D_1 \cup D_4$. Then, since $\{s, t\} \cap (N_G[1] \cup N_G[p]) = \emptyset$ and $L_1 \cup L_2 \cup L_4 \subset N[V_p]$, there exist two vertices (s, j_1) and (t, j_2) in L_5 such that j is adjacent to both j_1 and j_2 in H, which contradicts the fact that $\{\alpha_1, \ldots, \alpha_\theta\}$ is a covering of B_1.

Case 2.2: $|U_j \cap (D_1 \cup D_4)| = 1$.

Denote $U_j \cap (D_1 \cup D_4) = \{(i, j)\}$. If $(i, j) \in L_5$, then $j \in \phi_H(A_5)$ and $|U_j \cap (D_1 \cup D_4)| = 2$, a contradiction. Thus $(i, j) \notin L_5$. If $(i, j) \in A_1 \cup A_5$, then $U_j \cap N[D_1 \cup D_4] \subseteq N[(i, j), (p, j)]$. If $(i, j) \in L_1 \cup L_4$, then $U_j \cap N[D_1 \cup D_4] \subseteq N[(1, j), (i, j)]$. If $(i, j) \in L_2$, then $(q, j) \in U_j - N[D_1 \cup D_4]$. Thus we have that $U_j - N[D_1 \cup D_4] \neq \emptyset$.

Case 2.3: $|U_j \cap (D_1 \cup D_4)| = 2$.

In this case, $j \in (\phi_H(A_1) \cap \phi_H(L_1)) \cup (\phi_H(A_1) \cap \phi_H(L_4)) \cup (\phi_H(A_5) \cap \phi_H(L_1)) \cup (\phi_H(A_5) \cap \phi_H(L_5))$. It is easy to verify that $U_j \cap N[D_1 \cup D_4] \subseteq N[U_j \cap (D_1 \cup D_4)]$. Thus we have that $U_j - N[D_1 \cup D_4] \neq \emptyset$, since $\gamma(G) = 3$.

Thus, for each $j = 1, 2, \ldots, n$, there exists a vertex (y_j, j) in U_j such that (y_j, j) is not dominated by $D_1 \cup D_4$. Denote $W_2 = \{(y_j, j) \in U_j - N[D_1 \cup D_4] : j = 1, 2, \ldots, n\}$. Since W_2 is H-full and W_2 must be dominated by $D_5 = D - D_1 - D_4$, we have that $|D_5| \geq \gamma(H)$. Therefore, $|D| = |D_1| + |D_4| + |D_5| \geq 3\gamma(H)$. □

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